ECEN 5738 - Existence, Uniqueness, and Stability of Parabolic Partial Differential Equations

Andy Gusty, Undergraduate CU-Boulder, Dept of Applied Mathematics
YuKang Kong, Undergraduate CU-Boulder, Dept of Aerospace Engineering

August 26, 2025

Contents

1 Introduction

Partial differential equations (PDEs) are fundamental in modeling a wide variety of time-dependent physical, biological, and engineering systems. Of particular interest in both theory and applications are **parabolic PDEs**, which describe diffusive or dissipative processes such as heat conduction, fluid flow, and signal propagation in neurons. These equations are characterized by a time derivative of first order and spatial derivatives of second order, capturing both the evolution and smoothing behavior of the system over time.

In general form, a nonlinear parabolic PDE can often be expressed as an abstract evolution equation in a Banach space X,

$$\frac{d}{dt}\mathbf{u}(t) + A\mathbf{u}(t) = f(t, \mathbf{u}(t)), \tag{1}$$

where $\mathbf{u}(t) \in X$ denotes the state of the system at time t, A is a linear differential operator in the spatial variables (but independent of time), and $f(t, \mathbf{u})$ is a possibly nonlinear forcing term. The operator A typically involves partial derivatives of order up to 2 with respect to spatial variables, but does not involve time derivatives. A canonical example is the Laplace operator,

$$A = \Delta = \sum_{i=1}^{n} \partial_{x_i x_i}, \tag{2}$$

which models diffusion in n spatial dimensions.

Parabolic PDEs encompass a wide range of systems. Notable examples include:

- **The Heat Equation:** Models the flow of heat in a medium.
- **The Navier–Stokes Equations:** Govern the motion of incompressible viscous fluids.
- **The Hodgkin–Huxley Model:** Describes electrical conduction in nerve axons.

The goal of this paper is to investigate the well-posedness and stability of solutions to the ODE given in (1). For existence and uniqueness of solutions, we will focus on the linear version of (1) ($f(t, \mathbf{u}(t)) = 0$), but the theory shown in this paper lays the groundwork for investigating the nonlinear version (1) as well (see [1], Ch. 3.3). To accomplish this, we rely heavily on the theory developed by Dan Henry and Gabe Buis in [1] and [2] respectively, which enables the treatment of existence, uniqueness, and stability for a broad class of nonlinear evolution equations like (1) by interpreting the linear part A as the infinitesimal generator of a semigroup on a Banach space.

2 Preliminaries

Definition 1. *Let X be a vector space of the field of real or complex numbers. A* ***norm*** *on X, denoted by $\|\cdot\|$, is a real-valued function on X which satisfies,*

(i) $\|x\| \geq 0 \ \ \forall x \in X$

(ii) $x \neq 0 \implies \|x\| \neq 0$

(iii) $\|ax\| = |a| \, \|x\|$, *for a some scalar*

(iv) $\|x + y\| \leq \|x\| + \|y\|$ *triangle inequality*

The vector space X, together with the norm on X, is called a ***normed linear space***

Definition 2. *A **Banach space** X, is a normed linear space in which every Cauchy sequence converges with respect to the norm to a limit point in X.*

Definition 3. *Let X and Y be normed linear spaces and let $L : X \to Y$ define a linear operator. L is **bounded** if there exists a constant $c > 0$ such that,*

$$\|Lx\| \leq c\,\|x\|\,, \quad \forall x \in X \tag{3}$$

Definition 4. *For a linear operator $L : X \to Y$, where X, Y are normed linear spaces, L is **closed** if for each sequence $\{x_n\} \subset X$, if $x_n \to x$ and $Lx_n \to y$, then $x \in X$ and $Lx = y$.*

Definition 5. *For a linear operator $L : D(L) \subseteq X \to Y$, where X, Y are normed linear spaces, L is a **densely defined operator** if $D(L)$ is dense in X.*

2.1 Differential Operators

For a matrix-valued ODE of the form,

$$\dot{x} = Ax \tag{4}$$

Where $x \in X$, some n-dimensional Euclidean space, say $\mathbb{R}^n$, then A is a linear operator from X onto X, and thus A is bounded, i.e. there exists som $M > 0$ s.t.

$$\|Ax\| \leq M\|x\| \tag{5}$$

For any $x \in X$. But if X is a more general normed linear space, say $L^2(0,1)$ (square-integrable functions over the interval $(0,1)$), then since L^2 is infinite dimensional, meaning that its orthonormal basis is an infinite combination of basis functions - typically represented by trigonometric functions in the case of Fourier analysis. Functions are reconstructed via infinite series expansions, not finite sums, so A might be unbounded. As an informal example, we might have,

$$\frac{d}{dx}\sum_{n=1}^{\infty} a_n \sin(nx) = \sum_{n=1}^{\infty} n a_n \cos(nx), \tag{6}$$

which is unbounded since $n \to \infty$. In this case where X is a general Banach space, we should rewrite (5) as,

$$\frac{dx}{dt} = Ax, \quad (x \in D(A) \subseteq X \ ; \ A : D(A) \to R(A) \subseteq X) \tag{7}$$

We might be tempted to solve (7) using the exponential e^{At}, but this notion is not well-defined when A is unbounded. Luckily, a mathematical theory exists in which we can treat solutions as one-parameter families of bounded linear operators, which we call groups and semigroups.

3 Sectorial Operators and Semigroup Theory

3.1 Resolvent Operator

Definition 6. *$\mathcal{L}(X, Y)$ is the space of continuous linear operators with domain X and rage Y, where X, Y are both Banach spaces. For a linear operator $L : X \to Y \in \mathcal{L}(X, Y)$, we denote,*

(i) $D(L) =$ *range of* L

(ii) $R(L) =$ *domain of* L

(iii) $N(L) =$ *nullspace of* L $(N(L) = \{x \in X;\ Lx = 0\}$

(iv) $\sigma(L) = P\sigma(L) \cup C\sigma(L) \cup R\sigma(L) =$ *spectrum of* L*, we define in greater detail below.*

In the case of a square matrix, $A \in \mathbb{R}^{n\times n}$, the spectrum of A is given by its eigenvalues λ. We seek to satisfy the equation,

$$\begin{aligned} &Ax = \lambda x \\ \implies &(A - \lambda I)x = 0 \end{aligned} \tag{8}$$

If the mapping : $R^{n\times n} \to R^n$ is bijective (one-to-one and onto), then $(A - \lambda I)$ is invertible and has inverse $(A - \lambda I)^{-1}$, which means (8) has only the trivial solution $(x = 0)$. Analogously, for a linear operator $L : X \to Y$, where X, Y are Banach spaces, we define the **resolvent set** $\rho(L)$ to be send of complex numbers λ such that $(L - \lambda I) : X \to Y$ is bijective. The complement of the resolvent set is the **spectrum** of L, $\sigma(L) = \mathbb{C}\backslash\rho(L)$. The spectrum can be written as the union of three sets,

1. The *pointwise spectrum* of L, $P\sigma(L)$ is the set of $\lambda \in \sigma(L)$ s.t. $L - \lambda I : X \to Y$ is not one-to-one.
2. The *continuous spectrum*, $C\sigma(L)$, is the collection of $\lambda \in \sigma$ such that $L - \lambda I : X \to Y$ is one-to-one not onto and $R(L - \lambda I)$ is dense in Y.
3. The *residual spectrum*, $R\sigma(L)$, is the collection of $\lambda \in \sigma$ such that $L - \lambda I : X \to Y$ is one-to-one not onto and $R(L - \lambda I)$ is not dense in Y.

Definition 7. *If* $\lambda \in \rho(A)$*, the resolvent operator* R_λ*:* $X \to X$ *is defined by,*

$$R_\lambda u := (\lambda I - A)^{-1} u \tag{9}$$

According to the Closed Graph Theorem [3], $R_\lambda : X \to D(A) \subseteq X$ *is a bounded linear operator. Further,*

$$AR_\lambda u = R_\lambda A u \quad \text{if } u \in D(A) \tag{10}$$

Lemma 1. *Properties of resolvent operators*
If λ*,* $\mu \in \rho(A)$ *we have*

$$R_\lambda - R_\mu = (\mu - \lambda) R_\lambda R_\mu \tag{11}$$

and

$$R_\lambda R_\mu = R_\mu R_\lambda \tag{12}$$

1. Proof. By definition, we have that,

$$R_\lambda(\lambda I - A) = R_\mu(\lambda I - A) = I \tag{13}$$

Thus, we can multiply $R_\lambda - R_\mu$ by the identity,

$$\begin{aligned} R_\lambda - R_\mu &= I R_\lambda - R_\mu I && (14)\\ &= R_\mu(\mu I - A)R_\lambda - R_\mu(\lambda I - A)R_\lambda && (15)\\ &= R_\mu(\mu - \lambda)R_\lambda && (16)\\ &= (\mu - \lambda)R_\mu R_\lambda && (17) \end{aligned}$$

We can prove the second property using the same idea:

$$\begin{aligned} R_\lambda - R_\mu &= R_\lambda I - I R_\mu && (18)\\ &= R_\lambda(\mu I - A)R_\mu - R_\lambda(\lambda I - A)R_\mu && (19)\\ &= (\mu - \lambda)R_\lambda R_\mu && (20) \end{aligned}$$

Thus,

$$(\mu - \lambda)R_\lambda R_\mu = (\mu - \lambda)R_\mu R_\lambda \implies R_\lambda R_\mu = R_\mu R_\lambda \tag{21}$$

3.2 Semigroup Theory

Consider the following ordinary differential equation initial value problem (IVP),

$$\begin{cases} \frac{d\mathbf{u}}{dt} = A\mathbf{u}(t) \quad (t \geq 0) \\ \mathbf{u}(0) = u \end{cases} \tag{22}$$

Where $u \in X$ a Banach space, and $A : D(A) \to X$, is a linear partial differential operator involving variables other than t, with $D(A) \subseteq X$. Before we can investigate stability of solutions to (22), we must first determine the existence and uniqueness of a solution

$$\mathbf{u}(t) : [0, \infty) \to X \tag{23}$$

To write PDEs in this form, we have in mind that X is an L^p space of functions. Thus, $\mathbf{u}(t)$ maps values of t to functions of the other variables of interest. Let us suppose that we seek a solution of the form,

$$\mathbf{u}(t) := S(t)u \qquad (t \geq 0) \tag{24}$$

for $u \in X$ and $S(t) : X \to X$ is a linear operator. We require the following properties to be satisfied:

$$\lim_{t \to 0^+} S(t)u = u \qquad (u \in X) \tag{25}$$

$$S(t+s)u = S(t)S(s)u = S(s)S(t)u \qquad (t, s \geq 0,\ u \in X) \tag{26}$$

$$\text{the mapping } t \mapsto S(t)u \text{ is real analytic from } (0, \infty) \text{ into } X \tag{27}$$

Note that condition (26) is simply the assumption that (22) has a unique solutions for each initial point. In addition, (27) is often more restricted than needed for linear PDEs. However, enforcing analytical allows the treatment of certain classes of nonlinear PDEs as well (see [1], Ch. 3.3).

Definition 8. *An **analytic semigroup** on a Banach space X is a family of continuous linear operators on X, $\{S(t)\}_{t \geq 0}$, which satisfies (25), (26), and (27).*

Definition 9. *For a linear operator $A : D(A) \to X$, we define A to be the **infinitesimal generator** of the semigroup $\{S(t)\}_{t \geq 0}$ such that,*

$$D(A) := \left\{ u \in X \mid \lim_{t \to 0^+} \frac{S(t)u - u}{t} \textit{ exists in } X \right\} \tag{28}$$

$$Au := \lim_{t \to 0^+} \frac{S(t)u - u}{t} \tag{29}$$

It can be shown that for an infinitesimal generator A of an analytic semigroup, the domain $D(A)$ is dense in X, and that A is a closed operator [3]. We now prove a theorem which will allow us to develop a notion for a derivative of a semigroup.

Theorem 1. *Assume $u \in D(A)$, then,*

$$(i) \quad S(t)u \in D(A) \tag{30}$$

$$(ii) \quad AS(t)u = S(t)Au \quad \forall t \geq 0 \tag{31}$$

$$(iii) \quad \textit{The mapping } t \mapsto S(t)u \textit{ is differentiable for each } t > 0 \tag{32}$$

$$(iv) \quad \frac{d}{dt}S(t)u = AS(t)u \quad (t > 0) \tag{33}$$

Proof.
To prove (i) and (ii), let $u \in D(A)$, then

$$AS(t)u = \lim_{s\to 0^+} \frac{S(s)S(t)u - S(t)u}{s} \tag{34}$$
$$= \lim_{s\to 0^+} \frac{S(t)S(s)u - S(t)u}{s} \qquad \text{by (26)} \tag{35}$$
$$= S(t) \lim_{s\to 0^+} \frac{S(s)u - u}{s} = S(t)Au \tag{36}$$

Thus assertions (i) and (ii) are proved. Next, let $u \in D(A)$, $h > 0$. Then if $t > 0$,

$$\lim_{h\to 0+} \left\{ \frac{S(t)u - S(t-h)u}{h} - S(t)Au \right\} \tag{37}$$
$$= \lim_{h\to 0+} \left\{ \frac{S(t-h+h)u - S(t-h)u}{h} - (S(t) + S(t-h) - S(t-h)Au \right\} \tag{38}$$
$$= \lim_{h\to 0+} \left\{ \frac{S(t-h)S(h)u - S(t-h)u}{h} - (S(t) + S(t-h) - S(t-h))Au \right\} \tag{39}$$
$$= \lim_{h\to 0+} \left\{ S(t-h) \left(\frac{S(h)u - u}{h} - Au \right) + (S(t-h) - S(t))Au \right\} \tag{40}$$
$$= \lim_{h\to 0+} \left\{ S(t-h) \left(\frac{S(h)u - u}{h} - Au \right) + (S(t-h) - S(t))Au \right\} \tag{41}$$
$$= S(t)\left(Au - Au\right) + (S(t) - S(t))Au = 0 \tag{42}$$

Thus, $\frac{d}{dt}S(t)u$ exists for $t > 0$ and is equal to $S(t)Au = AS(t)u$, which proves (iii) and (iv). QED

3.3 Sectorial Operators

Definition 10. *A linear operator A in a Banach space X is a* ***sectorial operator*** *if it is a closed densely defined operator such that, for some ϕ in $(0, \pi/2)$ and some $M \geq 1$, the sector,*

$$S_\phi = \{\lambda;\ \phi \leq |arg(\lambda)| \leq \pi, \quad \lambda \neq 0\} \tag{43}$$

is in the resolvent set of A and,

$$\left\|(\lambda - A)^{-1}\right\| \leq \frac{M}{|\lambda|}, \qquad \forall \lambda \in S_\phi \tag{44}$$

Theorem 2. *If A is a sectorial operator, then $-A$ is the infintesimal generator of an analytic semigroup $\{e^{-tA}\}_{t\geq 0}$, where*

$$e^{-At} = \frac{1}{2\pi i} \int_\Gamma (\lambda + A)^{-1} e^{\lambda t} d\lambda \tag{45}$$

Where Γ is a contour in $\rho(-A)$ with $arg(\lambda) \to_{|\lambda|\to\infty} \pm\theta$ for some $\theta \in (\pi/2, \pi)$

Further, there exists some constant C such that for $Re(\lambda) > 0$ whenever $\lambda \in \sigma(A)$ for $t > 0$,

$$\left\|e^{-At}\right\| \leq Ce^{-At} \tag{46}$$

Finally,

$$\frac{d}{dt}e^{-At} = -Ae^{-At} \tag{47}$$

Proof. Let $R_\lambda = (\lambda + A)^{-1}$ denote the resolvent operator of $-A$. Choose $\theta \in (\pi/2, \pi - \phi)$, define e^{-At} by the above integral. Γ lies in the left-half plane and $-A$ is sectorial with respect to the angle $\pi - \phi$. Thus, the bound in (44) holds for R_λ. We then consider t in the compact set $\{|arg(t)| \leq \epsilon\}$ for a sufficiently small $\epsilon > 0$. With this established, we can use the Weierstrass M-test:

$$\left\| e^{\lambda t} R_\lambda \right\| \leq e^{-c|\lambda||t|} \frac{M}{|\lambda|} \tag{48}$$

to show that the integral it converges absolutely if $t > 0$. Note that $\Re(\lambda) < 0, \quad t > 0, \quad c > 0$ along $\Gamma \to \infty$. Because of this, the semigroup is analytic in any compact set of $\{|arg(t)| \leq \epsilon\}$, which shows condition (27) for an analytic semigroup (uniform convergence of the Dunford integral implies the semigroup is analytic [4]).

We now show that this satisfies (26) (Semigroup property). By Cauchy's Theorem, the integral is unchanged when the contour Γ is shifted to the right a small distance, we will call this new contour Γ'. Then for $t > 0$, $s > 0$,

$$\begin{aligned} e^{-At}e^{-As} &= \frac{1}{(2\pi i)^2} \int_\Gamma R_\lambda e^{\lambda t} d\lambda \int_{\Gamma'} R_\mu e^{\mu s} d\mu && (49) \\ &= \frac{1}{(2\pi i)^2} \int_\Gamma \int_{\Gamma'} R_\lambda e^{\lambda t} R_\mu e^{\mu s} \, d\mu \, d\lambda && (50) \\ &= \frac{1}{(2\pi i)^2} \int_\Gamma \int_{\Gamma'} e^{\lambda t + \mu s} (\mu - \lambda)^{-1} (R_\lambda - R_\mu) \, d\mu \, d\lambda && (51) \end{aligned}$$

Where (51) follows by applying the resolvent identity from Lemma 1. This breaks the integral up in into two parts, we examine the second part first,

$$\begin{aligned} I_2 &= \frac{1}{(2\pi i)^2} \int_\Gamma \int_{\Gamma'} e^{\lambda t + \mu s} \frac{-R_\mu}{\mu - \lambda} d\mu d\lambda && (52) \\ &= -\frac{1}{(2\pi i)^2} \int_{\Gamma'} e^{\mu s} R_\mu \left(\int_\Gamma \frac{e^{\lambda t}}{\mu - \lambda} d\lambda \right) d\mu && (53) \end{aligned}$$

Note that switching order of integration above is valid by Fubini's Theorem since the integral converges absolutely. Further, since the integrand $(\mu - \lambda)^{-1} e^{\lambda t}$ is analytic as long as $\mu \neq \lambda$. Since $\mu \in \Gamma'$ lies just outside $\lambda \in \Gamma$, the integrand is analytic on and inside Γ. Additionally, $\lambda \in \rho(-A) \implies Re(\lambda) \to -\infty$ along Γ in the left-half plane. Thus by Cauchy's Theorem,

$$I_2 = 0. \tag{54}$$

We now focus on the remaining integral,

$$\begin{aligned} I_1 &= \frac{1}{(2\pi i)^2} \int_\Gamma \int_{\Gamma'} e^{\lambda t + \mu s} R_\lambda (\mu - \lambda)^{-1} d\mu d\lambda && (55) \\ &= \frac{1}{(2\pi i)^2} \int_\Gamma e^{\lambda t} R_\lambda \left(\int_{\Gamma'} \frac{e^{\mu s}}{\mu - \lambda} d\mu \right) \, d\lambda && (56) \end{aligned}$$

The integrand of the inner integral $(e^{\mu s}(\mu - \lambda)^{-1}$ is analytic everywhere except the simple pole at $\mu = \lambda$ inside the contour Γ'. By Cauchy's Residue Theorem [5],

$$\begin{aligned} \int_{\Gamma'} \frac{e^{\mu s}}{\mu - \lambda} d\mu &= 2\pi i (e^{\mu s})|_{\mu = \lambda} && (57) \\ &= 2\pi i e^{\lambda s} && (58) \end{aligned}$$

Thus,

$$I_1 = \frac{1}{2\pi i}\int_\Gamma e^{\lambda(t+s)}R_\lambda d\lambda = e^{-A(t+s)} \tag{59}$$

This establishes the semigroup property $e^{-A(t+s)} = e^{-At}e^{-As}$ (which is the condition of (26)) (semigroup property). We now have to show that (25) (initial condition requirement) is satisfied, i.e. $e^{-At}x \to_{t\to 0^+} x$ for each $x \in X$.

$$e^{-At}x - x = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\left(R_\lambda - \lambda^{-1}\right)xd\lambda = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}R_\lambda\lambda^{-1}\left(\lambda I - (\lambda I - A)\right)xd\lambda \tag{60}$$

$$\text{and,} \qquad \left\|\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\lambda^{-1}AR_\lambda xd\lambda\right\| \leq \frac{M\left\|Ax\right\|}{2\pi}\int_\Gamma |e^{\lambda t}||\lambda|^{-2}|d\lambda| \tag{61}$$

$$\leq Ct\left\|Ax\right\| \qquad \text{For some constant } C \tag{62}$$

Which is justified by the fact that $x = e^{0t}x$ can be written as a Dunford integral,

$$x = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(I\lambda - 0)^{-1}xd\lambda = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\lambda^{-1}xd\lambda \tag{63}$$

Thus, (25) is satisfied. Next, for $\Re(\lambda) < a$, $\lambda \in \Gamma$, we perform a u-substitution $u = \lambda t$, to derive

$$\left\|e^{-At}\right\| = \left\|\frac{1}{2\pi i}\int_\Gamma e^u(\frac{u}{t} + A)^{-1}\frac{du}{t}\right\| \leq \frac{M}{2\pi}\int_\Gamma |e^u|\frac{|du|}{|u|} \leq Ce^{-at} \tag{64}$$

The final thing to show is that $\frac{d}{dt}e^{-At} = -Ae^{-At}$:

$$\frac{d}{dt}e^{-At}x + Ae^{-At}x = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda + A)(\lambda + A)^{-1}xd\lambda = 0, \tag{65}$$

again by Cauchy's Theorem since the integrand is analytic every on and inside Γ. QED.

3.3.1 Existence and Uniqueness

Let us briefly return to the homogeneous problem

$$\begin{aligned} &\frac{d\mathbf{u}}{dt} + A\mathbf{u} = 0, \quad t > 0 \\ &\mathbf{u}(0) = u_0, \end{aligned} \tag{66}$$

where A is a sectorial operator in a Banach space X and $u \in X$ is given. A solution of (66) on $0 < t < T$ is a continuous function $\mathbf{u} : [0, T) \to X$ which is continuously differentiable on the open interval $(0, T)$, has $\mathbf{u}(t) \in D(A)$ for $0 < t < T$, and satisfies (66) on $(0, T)$ with $\mathbf{u}(t) \to_{t\to 0^+} u_0$. Based on the previous analysis, it is obvious that $\mathbf{u} = e^{-At}u_0$ is a valid solution of (66), we now prove it is unique.

Let $0 \leq s \leq t < T$ and

$$y(t, s) := e^{-A(t-s)}\mathbf{u}(s), \tag{67}$$

where $\mathbf{u}(s)$ is any solution of (66) on $(0, T)$. Then $s \mapsto t(t, s$ is continuous on $s \in [0, t]$ and continuous differentiable on $s \in (0, t)$ with

$$\frac{\partial y(t,s)}{\partial s} = e^{-A(t-s)}\frac{d\mathbf{u}(s)}{(ds)} + Ae^{-A(t-s)}\mathbf{u}(s) = 0 \tag{68}$$

for $0 < s < t$. This implies that for fixed t, $y(t, \cdot)$ is a constant, i.e. $y(t, 0) = y(t, s) \quad \forall s \in (0, t)$. Thus, since $\lim_{s\to t} y(t, s) = \mathbf{u}(t)$

$$e^{-At}u_0 = \mathbf{u}(t) \tag{69}$$

3.4 Walkthrough of 1-D Heat Equation

So far, everything we have covered has been very abstract, so we now walk through applying semigroup theory to a simple PDE: the one-dimensional heat equation with Neumann boundary conditions. Let U be the bounded open interval over the real line of length L, i.e. $U = (0, L)$. We have the following problem:

$$\begin{cases} \partial_t u - \partial_{xx} u = 0, & u \in U \times [0, T] \\ \partial_x u = 0, & \text{on } \partial U \times [0, T] \\ u = u_0, & \text{on } U \times \{t = 0\} \end{cases} \tag{70}$$

We now propose to reinterpret (70) as the flow determined by an analytic semigroup on $X = L^2(U)$. For this, we set,

$$D(A) := H_0^1(U) \cap H^2(U) \tag{71}$$

and define

$$Au := -\partial_{xx} u \tag{72}$$

First, we seek to show that A is sectorial. Note that $C_0^\infty(U) \subset Dom(A)$ is dense in $L^2(U)$, so A is densely defined.

We also claim that A is closed. Let $\{u_k\}_{k=1}^\infty \subset D(A)$ with,

$$u_k \to u, \quad Au_k \to v \quad \text{ in } L^2(U). \tag{73}$$

According to Theorem 4 of 6.3.2 of [3],

$$\|u_k - u_l\|_{H^2(U)} \leq C(\|Au_k - Au_l\|_{L^2(U)} + \|u_k - u_l\|_{L^2(U)}). \tag{74}$$

By assumption, both sequences on the RHS converge in $L^2(U)$, which implies $\{u_k\}_{k=1}^\infty$ is a Cauchy sequence in $H^2(U)$ and so,

$$u_k \to u \quad \text{ in } H^2(U), \tag{75}$$

thus $u \in D(A)$. Further, this also implies $Au_k \to Au \in L^2(U)$, with $v = Au$. Thus, A is closed.

We now check the resolvent conditions for a sectorial operator. To find the spectrum of A, we solve the eigenvalue problem,

$$-\partial_{xx} \phi_n = \lambda_n \phi_n, \quad \phi_n \in D(A), \quad \lambda_n \in \mathbb{C}, \tag{76}$$

which for our boundary conditions has nontrivial solutions,

$$\phi_n = \cos(\frac{n\pi}{L} x), \quad \lambda_n = (\frac{n\pi}{L})^2, \quad n = 0, 1, 2, ... \tag{77}$$

Thus, $\sigma(A) \subset [0, \infty)$, which implies the sector, S_ϕ defined in (43) is in $\rho(A)$ for any $\phi \in (0, \pi/2)$. Finally, we need to show that (44) holds for A. Consider the equation following equation for some $f : D(A) \to L^2(U)$,

$$(\lambda I - A)u = f, \quad \frac{du}{dx}(0) = \frac{du}{dx}(L) = 0, \tag{78}$$

for some $f \in L^2(0, L)$ and $\lambda \in S_\phi$.

Multiplying both sides by u and integrating over $U = (0, L)$, we obtain

$$\langle \lambda u - u_{xx}, u \rangle = \langle f, u \rangle. \tag{79}$$

Using integration by parts and the Neumann boundary conditions, we find

$$\lambda\|u\|_{L^2}^2 + \left\|\frac{du}{dx}\right\|_{L^2}^2 = \langle f, u\rangle. \tag{80}$$

Taking the real part of both sides yields

$$\Re(\lambda)\|u\|_{L^2}^2 + \left\|\frac{du}{dx}\right\|_{L^2}^2 \leq \|f\|_{L^2}\|u\|_{L^2}. \tag{81}$$

Assuming $\lambda \in S_\phi$ with $\phi < \frac{\pi}{2}$, we have $\Re(\lambda) \geq c|\lambda|$ for some constant $c > 0$, and hence

$$\|u\|_{L^2} \leq \frac{\|f\|_{L^2}}{\Re(\lambda)} \leq \frac{C}{|\lambda|}\|f\|_{L^2}. \tag{82}$$

This implies

$$\|(\lambda I - A)^{-1} f\|_{L^2} \leq \frac{C}{|\lambda|}\|f\|_{L^2}, \tag{83}$$

and therefore

$$\boxed{\|(\lambda I - A)^{-1}\|_{\mathcal{L}(L^2)} \leq \frac{C}{|\lambda|}}, \tag{84}$$

for all $\lambda \in S_\phi$ with $|\lambda|$ sufficiently large. This establishes the resolvent estimate required for A to be sectorial. Since A is sectorial, $-A$ generates a bounded analytic semigroup $\{e^{-At}\}_{t\geq 0}$. The mild solution is thus given by

$$u(t) = e^{-At}u_0. \tag{85}$$

The semigroup e^{-At} admits the Dunford–Taylor integral representation:

$$e^{-At} = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda I + A)^{-1}\, d\lambda, \tag{86}$$

where Γ is a contour in the complex plane enclosing the spectrum of A in the sector S_ϕ, oriented counter-clockwise.

Applying this operator to the initial condition $u_0 \in L^2(0, L)$, we obtain

$$u(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda + A)^{-1}u_0\, d\lambda. \tag{87}$$

Next, we expand u_0 in the orthonormal eigenbasis $\{\phi_n\}_{n=0}^\infty$ of A:

$$u_0(x) = \sum_{n=0}^\infty a_n\phi_n(x), \quad a_n = \langle u_0, \phi_n\rangle, \tag{88}$$

where

$$\phi_0(x) = a_0, \quad \phi_n(x) = a_n\cos\left(\frac{n\pi x}{L}\right), \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2.$$

For a_0, a_n some constants. The resolvent operator acts diagonally on this basis (note that the resolvent operator is the Laplace transform of the semigroup [3]):

$$(\lambda + A)^{-1}u_0 = \sum_{n=0}^\infty \frac{a_n}{\lambda + \lambda_n}\phi_n.$$

Substituting into the integral yields:

$$u(t) = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} \sum_{n=0}^{\infty} \frac{a_n}{\lambda + \lambda_n} \phi_n(x) \, d\lambda \tag{89}$$

$$= \sum_{n=0}^{\infty} a_n \phi_n(x) \left(\frac{1}{2\pi i} \int_\Gamma \frac{e^{\lambda t}}{\lambda + \lambda_n} \, d\lambda \right). \tag{90}$$

This integral is the inverse Laplace transform of $\lambda \mapsto \frac{1}{\lambda + \lambda_n}$, evaluated at time t. By the Cauchy residue theorem, since $\lambda_n \in \mathbb{R}^+ \subset \text{int}(\Gamma)$, we have:

$$\frac{1}{2\pi i} \int_\Gamma \frac{e^{\lambda t}}{\lambda + \lambda_n} \, d\lambda = e^{-\lambda_n t}. \tag{91}$$

Hence,

$$u(x,t) = \sum_{n=0}^{\infty} a_n e^{-\lambda_n t} \phi_n(x),$$

which is the classical solution to the heat equation via cosine series expansion.

4 Stability of Dynamical Systems

The motivation for converting a partial differential equation (PDE) into an abstract form involving an analytic semigroup is to leverage powerful theorems from dynamical systems theory—particularly those concerning stability. Since analytic semigroups form a special class of dynamical systems, results about the latter can be applied to analyze the former.

Definition 11. *A dynamic system on a complete metric space C is a family of maps $\{S(t) : C \to C, t \geq 0\}$ such that:*

1. *for each $t \geq 0$, $S(t)$ us continuous from C to C;*
2. *for each $x \in C$, $t \to S(t)x$ is continuous;*
3. *$S(0) =$ identity on C;*
4. *$S(t)(S(\tau)x) = S(t+\tau)x$ for all $u \in C$ and $t, \tau \geq 0$.*

From this definition, it is clear that an analytic semigroup satisfies all the properties of a dynamical system. Therefore, we can apply dynamical systems theorems—such as those concerning long-term behavior or stability—directly to analytic semigroups.

4.1 Notion of Stability

Before we can analyze the stability of a dynamical system, we must first define what stability means in this context.

Definition 12. *Let $\{S(t)u, t \geq 0\}$ be a dynamical system on C for any $u \in C$, let $\gamma(u) = \{S(t)u, t \geq 0\}$. u is an **equilibrium point** if $\gamma(u) = \{u\}$.*

Definition 13. *Let $\{S(t)u, t \geq 0\}$ be a dynamical system on C for any $u \in C$ let $\gamma(u) = \{S(t)u, t \geq 0\}$. $\gamma(u)$ is **stable** if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for all $t \geq 0$, $dist(S(t)u, S(t)v) < \varepsilon$ whenever $dist(u,v) < \delta(\epsilon)$, $v \in C$*

Definition 14. *Let $\{S(t)u, t \geq 0\}$ be a dynamical system on C for any $u \in C$ let $\gamma(u) = \{S(t)u, t \geq 0\}$ $\gamma(u)$ is **uniformly asymptotically stable** if it is stable and there is a neighborhood $V = \{v \in C; dist(u,v) < r\}$ such that $dist(S(t)v, S(t)u) \to 0$ as $t \to +\infty$, uniformly for $v \in V$*

4.2 Lyapunov Theorem

To study stability, we now introduce the concept of a Lyapunov function, which provides a powerful method for proving the stability of a dynamical system.

Theorem 3. *Let $\{S(t)\, t \geq 0\}$ be a dynamical system on C and let 0 be an equilibrium point in C. The equilibrium point 0 is* ***stable*** *if there exist a function $V : C \to \mathbb{R}$ and a class-$\mathcal{K}$ function $\alpha(\cdot)$ such that for all $t > 0$, $u \in C$:*

1. $V(0) = 0$
2. $V(u) \geq \alpha(||u||)$
3. $\dot{V}(u) \leq 0$

Proof. For each $k > 0$, define set $U_k = \{u \in C; V(u) < k\}$, which is neighborhood of 0. As $\dot{V}(u) \leq 0$, U_k is positive invariant: $u \in U_k$ implies $V(S(t)u) \leq V(u) \leq k$ for all $t \geq 0$. If $V(u) \geq \alpha(||u||)$, then for any $\varepsilon > 0$, there exists $k = \alpha(\varepsilon) > 0$ such that $\alpha(\varepsilon) = k \geq V(u) \geq \alpha(||u||) \Rightarrow \varepsilon \geq ||u||$. And by continuity of V, we can always find $||u|| < \delta$ such that $S(t)u \in U_k$ for all $t \geq 0$. And thus $||S(t)u|| < \epsilon$ for all $t > 0$. □

Theorem 4. *Let $\{S(t)\, t \geq 0\}$ be a dynamical system on C and let 0 be an equilibrium point in C. The equilibrium point 0 is* ***uniformly asymptotically stable*** *if there exist a function $V : C \to \mathbb{R}$ and two class-$\mathcal{K}$ function α_1, α_2 such that for all $t > 0$, $u \in C$:*

1. $V(0) = 0$
2. $V(u) \geq \alpha_1(||u||)$
3. $\dot{V}(u) \leq -\alpha_2(||u||)$

Proof. Continue from the proof before, we know that $V(S(t)u)$ is a non-increasing function of t, bounded below by 0. Let as $t \to +\infty$, $V(S(t)u) \to l$. If $l > 0$, then $\inf_{t\geq 0} ||S(t)u||$ is positive and thus $\sup_{t\geq 0} \dot{V}(S(t)u) \leq -m$ for $m > 0$. However, this contradicts with the fact that $V(S(t)u) \geq 0$ for all $t \geq 0$. Thus $V(S(t)u$ and $||S(t)x||$ must go to zero as $t \to +\infty$. □

4.2.1 Exercise

Show that a flat wall, with thickness h and constant temperature on two side is asymptotically stable regardless of initial condition within the wall.

Formulated pde:

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \\ u(0,t) = u(h,t) = 0, \text{ for all } t \geq 0 \end{cases}$$

Let $V = \frac{1}{2}\int_0^h (u(x,t))^2 dx$ to be the Lyapunov function. It is obvious that this choice meet the first requirement as it will always be non negative. Furthermore, as we can rewrite V as $V = \frac{1}{2}(||u||_{L^2})^2$, it also satisfy the second requirement.

Now we take the time derivative of the Lypunov function.

$$\frac{dV}{dt} = \frac{1}{2}\int_0^h \frac{\partial}{\partial t} u^2 dx = \int_0^h u \frac{\partial u}{\partial t} d$$

By plugging in the pde of the heat transfer equation, we get

$$\frac{dV}{dt} = \int_0^h u \frac{\partial^2 u}{\partial x^2} dx$$

Using integration by part:

$$\frac{dV}{dt} = [u \cdot u_x]_0^h - \int_0^h u_x^2 dx = -\int_0^h u_x^2 dx = -||u||_{L^2}^2$$

With this, we already proof that the system is stable. To proof it to be uniformly asymptotically stable, we still need to show $\frac{dV}{dt} \leq -\alpha(||u||)$. To do that, we can use Poincaré inequality, which C is a constant:

$$||u||_{L^2} \leq C|||u_x||_{L^2}$$
$$-||u||_{L^2}^2 \geq -C^2|||u_x||_{L^2}^2$$

With this inequality, we can rewrite $\frac{dV}{dt}$ as:

$$\frac{dV}{dt} \leq -\frac{1}{C^2}||u||_{L^2}^2$$

And thus show that it is indeed uniformly asymptotically stable.

4.3 Invariance Principle

Although Lyapunov's Theorem is a powerful tool for proving stability, it is often difficult in practice to construct a Lyapunov function that satisfies all the required conditions.

In such cases, the Invariance Principle can serve as a useful alternative. While it does not always establish stability directly in the same way a Lyapunov function does, it can still provide meaningful insights into the asymptotic behavior of a dynamical system.

Definition 15. *If $u_0 \in C$, then the **ω-limit set** for u_0 is*

$$\omega(u_0) = \{u \in C \text{ — there exist } t_n \to \infty \text{ such that } S(t_n)u_0 \to u\}.$$

Theorem 5. *Suppose $u_0 \in C$ and $\{S(t)u_0, t \geq 0\}$ lies in a compact set in C; then $\omega(x)$ is nonempty, compact, invariant, connected, and $dist(S(t)u_0, \omega(u_0)) \to 0$ as $t \to 0$.*

Theorem 6. *Let V be a Lyapunov function on C and define $E = \{u \in C; \dot{V}(x) = 0\}$, M = maximal invariant subset of E. If $\{X(t)u_0, t \geq 0\}$ lies in a compact set in C, then $S(t)u_0 \to M$ as $t \to +\infty$.*

Theorem 7. *Suppose A is sectorial in X and has compact resolvent, and suppose U is an open set in $X^\alpha, \alpha < 1$, and $f : \mathbb{R}^+ \times U \to X$ is locally Lipchitz and $f(\mathbb{R}^+ \times B)$ is bounded in X for any closed bounded $B \subset U$. Finally assume $||f(t,u) - g(u)|| \to 0$ as $t \to \infty$, uniformly in a neighborhood of each $u \in U$, and $g(\cdot)$ is locally Lipchitz in U.*

Then if $\frac{du}{dt} + Au = f(t,u)$ for $t \geq t_0 \geq 0$ has a solution $u(\cdot)$ which is a closed bounded set $B \subset U$ on $t_0 \leq t \leq \infty$, then

$$dist_{X^\alpha}\{x(t), M\} \to 0 \text{ as } t \to \infty,$$

where M is the maximal invariant subset of B for $\frac{dv}{dt} + Av = f(t,v)$

5 Conclusion

In this paper, we used semigroup theory to convert parabolic partial differential equations into abstract ordinary differential equations which describe the evolution of a parabolic system from initial data. We used this theory, along with some complex functional analysis, to define the notion of an operator exponential for sectorial operators, and applied this to a general linear, homogeneous parabolic PDE to guarantee existence and uniqueness of solutions. Finally, we interpret these semigroups as dynamic systems to analyze the stability of solutions, and show that the 1-D Heat Equation with Dirichlet boundary conditions has a uniformly asymptotically stable equilibrium solution.

These results generalize many of the ideas we learned in class about existence, uniqueness, and stability of first order ODEs to a broader class of systems which have dependence on multiple variables. Future work will focus on applying analytic semigroups to nonlinear parabolic PDEs, to guarantee well-posedness and characterize solutions. Additionally, we would like to study the applications of stability and semigroup theory to hyperbolic and elliptic PDEs, in particular nonlinear wave equations.

References

[1] Daniel Henry. *Geometric Theory of Semilinear Parabolic Equations*, volume 840 of *Lecture Notes in Mathematics*. Springer-Verlag, 1981.

[2] Gabe Buis. Stability of partial differential equations. Unpublished manuscript, CU Boulder, 2024.

[3] Lawrence C. Evans. *Partial Differential Equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, 2nd edition, 2010.

[4] Einar Hille and Ralph S. Phillips. *Functional Analysis and Semi-Groups*, volume 31 of *Colloquium Publications*. American Mathematical Society, rev. ed. edition, 1957.

[5] Mark J. Ablowitz and Athanassios S. Fokas. *Complex Variables: Introduction and Applications*. Cambridge University Press, 2nd edition, 2003.